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Uniqueness of Solutions to a Two-Dimensional Mean Problem

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Let $0 < r_m < 1$, $r_m{}^m \leqslant \rho$ for all large m, and let $w_n = e^{i2\pi/n}$, $n = 1, 2,\ldots$. For a function $f(z) = \sum a_n z^n$, holomorphic in the open unit disk U, let $s_n(f) = (1/n) \sum_{k=1}^{n} f(r_n w_n{}^k)$, the nth arithmetic mean of f over the circle $|z| = r_n$. We prove that if $\rho < 1$ and $a_n = 0(n^{-\alpha_1})$ for $\alpha_1 = 1.728\ldots$, then f is uniquely determined by the two-dimensional means $s_n(f)$, $n = 1, 2,\ldots$. We also prove that for each ρ, $0 < \rho < 1$, there is a nontrivial f, holomorphic in U, such that $s_n(f) = 0$ for $n = 1, 2,\ldots$ with $r_n = \rho^{1/n}$.

1. INTRODUCTION AND RESULTS

Let U denote the open unit disk $|z| < 1$ in the complex plane and H the space of functions holomorphic in U. Let $0 < r_n < 1$, $n = 1, 2,\ldots$, and consider the means

$$s_n(f) = \frac{1}{n} \sum_{k=1}^{n} f(r_n e^{i2\pi k/n}) \tag{1.1}$$

of $f \in H$ on the concentric circles $|z| = r_n$. In this note, we study the problem of uniqueness of f when $s_n(f)$, $n = 1, 2,\ldots$, are given. This problem was posed in [2] and discussed in [1]. It was proved, in particular, that if $r_n{}^n \leqslant \rho$, $\rho \leqslant \frac{1}{2}$, for all n, and $f(z) = \sum a_n z^n$ with $\sum |a_n| < \infty$, then f is uniquely determined by the sequence $s_n(f)$, $n = 1, 2,\ldots$. The condition $\rho \leqslant \frac{1}{2}$ was a technical one. Here, by using a different method we prove a uniqueness result for any ρ, $0 < \rho < 1$. Of course a "smoothness" condition on f is required. In fact, we also obtain a negative result for each ρ, $0 < \rho < 1$. We state our main results in the following theorems.

THEOREM 1. *Let $0 < r_n < 1$ with $r_n{}^n \leqslant \rho < 1$ for all large n. Then there exists an α_ρ, $1 \leqslant \alpha_\rho < \alpha_1 = 1.728\ldots$, such that any function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, satisfying $a_n = 0(n^{-\alpha_\rho})$ and $s_n(f) = 0$ for $n = 1, 2,\ldots$, must be identically zero.*

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In the above theorem, α_ρ is uniquely determined by $\phi(\rho, \alpha_\rho) = 2\rho$, where

$$\phi(z, s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \tag{1.2}$$

is the so-called polylogarithm function (cf. [3, 4]). In particular, α_1 satisfies $\zeta(\alpha_1) = 2$ where ζ is the Riemann zeta-function. Calculation gives $\alpha_1 = 1.728\ldots$. As a simple consequence of the above theorem, we have

COROLLARY 1. *Let $0 < r_n < 1$ with $r_n{}^n \leqslant 0.79$ for all large n. Then any function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, satisfying $a_n = o(n^{-1})$ and $s_n(f) = 0$ for $n = 1, 2,\ldots$, must be the zero function.*

On the other hand, we have the following negative result.

THEOREM 2. *Let $0 < \rho < 1$ be given and $r_n = \rho^{1/n}$. Then there exists a complex number $\beta = \beta(\rho)$ such that the function $f(z) = \phi(z, \beta)$ satisfies $s_n(f) = 0$ for all n, $n = 1, 2,\ldots$.*

2. PROOF OF THE MAIN RESULTS

We first prove the following:

LEMMA. *Let $C = (c_{i,j})$, $i, j = 1, 2,\ldots$, be an upper triangular matrix with nonzero diagonal elements. Suppose that for some $\alpha \geqslant 0$,*

$$\sum_{j=k+1}^{\infty} | c_{k,j} | j^{-\alpha} \leqslant k^{-\alpha} | c_{k,k} |, \qquad k = 1, 2,\ldots . \tag{2.1}$$

Then for every $b = (b_1, b_2, \ldots)^T$ satisfying $b_n = o(n^{-\alpha})$ and $Cb = 0$, $b = 0$.

Proof. Suppose $b \neq 0$. Since $n^\alpha b_n \to 0$, there exists a k such that $k^\alpha | b_k |$ is maximum. We pick the largest such k, so that $k^\alpha | b_k | > n^\alpha | b_n |$ for all $n > k$. Hence, since $Cb = 0$ and $c_{k,k} \neq 0$, we have

$$| c_{k,k} | \, | b_k | \leqslant \sum_{j=k+1}^{\infty} | c_{k,j} | \, | b_j | < k^\alpha | b_k | \sum_{j=k+1}^{\infty} | c_{k,j} | j^{-\alpha}.$$

This is a contradiction to (2.1).

We can now prove Theorem 1. Without loss of generality, we assume that $r_n{}^n \leqslant \rho$ for all n. Let $f(z) = \sum a_n z^n$ be in H satisfying $a_n = o(n^{-\alpha_\rho})$ and $n = 1, 2,\ldots$. As in [1], it can easily be shown that $f(0) = a_0 = 0$, and

$$s_n(f) = \sum_{k=1}^{\infty} r_k^{nk} a_{nk} . \tag{2.2}$$

Hence, $a = (a_1, a_2, ...)^T$ satisfies the equation $Ca = 0$ with $C = (c_{k,j})$ and

$$\begin{aligned} c_{k,j} &= 0 \qquad \text{if } k \nmid j \\ &= r_k^{\,j} \qquad \text{if } k \mid j. \end{aligned}$$

Since $\phi(x, \alpha_\rho)/x$ is monotone increasing in x, and $r_k^{\,k} \leqslant \rho$ for all k, we have $\phi(r_k^{\,k}, \alpha_\rho)/r_k^{\,k} \leqslant \phi(\rho, \alpha_\rho)/\rho = 2$. Thus, we have

$$\sum_{t=2}^{\infty} r_k^{kt} t^{-\alpha_\rho} \leqslant r_k^{\,k}$$

or

$$\sum_{t=2}^{\infty} r_k^{kt} (kt)^{-\alpha_\rho} \leqslant k^{-\alpha_\rho} r_k^{\,k},$$

which is (2.1) with $\alpha = \alpha_\rho$. Hence, $a = 0$, or $f \equiv 0$, by the above lemma.

To prove Corollary 1, we observe that $\phi(x, 1)/x = -[\log(1 - x)]/x = 2$ for $x = 0.79....$ Hence, as above, if $r_n^{\,n} \leqslant 0.79$, then $\phi(r_n^{\,n}, 1) < 2r_n^{\,n}$, which gives (2.1) with $\alpha = 1$.

To prove Theorem 2, we set $r_n = \rho^{1/n}$ where ρ is any given positive number less than 1. For this fixed ρ, it was shown in [3] that the polylogarithm function $\phi(\rho, s)$ has many complex zeros. Let $\beta = \beta(\rho)$ be one of them, and define $f(z) = \phi(z, \beta)$. We have

$$s_n(f) = \sum_{t=1}^{\infty} \frac{\rho^t}{(nt)^\beta} = \frac{1}{n^\beta} \phi(\rho, \beta) = 0$$

for all $n = 1, 2, ...$

We remark that from [3], Re $\beta(\rho) < 1$ and we can choose $\beta(\rho)$ such that $\beta(\rho) \to 1$ as $\rho \to 1^-$.

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